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Observables in topological Yang–Mills theories with extended shift supersymmetry*

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Abstract. We present a complete classification, at the classical level, of the observables of topological Yang–Mills theories with an extended shift supersymmetry of N generators, in any space-time dimension. The observables are defined as the Yang–Mills BRST cohomology classes of shift supersymmetry invariants. These cohomology classes turn out to be solutions of an N-extension of Witten's equivariant cohomology. This work generalizes results known in the case of shift supersymmetry with a single generator.

1 Introduction

The prototype for topological theories of Witten's type is the four dimensional topological Yang–Mills theory of Witten [1–3], whose quantum observables are the Donaldson invariants [1, 4]. This model is characterized by a shift invariance, or "shift supersymmetry", generated by a single scalar fermionic charge, which is interpreted as the BRST invariance describing the non-physical character of the connection, with the result that only global "observables", namely the Donaldson invariants, are present. Generalizations to supersymmetry (SUSY) with $N = 2$ or more generators were already proposed some years ago in [5–7] and more recently in [8–11]. The construction of Lagrangian models for the gauge fixing of the shift supersymmetries may be found in [6] and, for arbitrary N and arbitrary space-time dimension, in [10, 11]. Most of these constructions are based on a superspace formalism for shift-SUSY introduced first in [12]. There, gauge invariance is formulated in terms of superfields. The Faddeev–Popov ghosts being superfields, supergauge invariance represents invariances with respect to a supermultiplet of local symmetries. However all these local invariances, except the original gauge invariance, may be fixed algebraically in a manner very similar to the Wess–Zumino (WZ) gauge fixing of the usual supersymmetric gauge theories [13]. The equivalence of the superspace theory and of the original one, e.g. that of Witten in the $N = 1$ case, is made explicit in this WZ-like gauge [6, 10, 14].

The problem of the characterization of the observables of such theories was defined by Witten [1, 15] as the computation of "equivariant cohomology", i.e. the cohomology of the shift-SUSY generator in the space of the gauge invariant local functionals of the fields. It was shown in [14] that this problem is almost equivalent to the presumably more tractable one of calculating, in the superspace formalism, the cohomology of the BRST operator associated with superspace gauge invariance, in the space of shift-SUSY invariant local functionals. In fact, this equivalence is exact up to solutions which are obviously trivial in the sense of Witten's equivariant cohomology.

Our purpose is to characterize, for general N and general space-time dimension, and in a formal classical set-up, all the observables defined as solutions of the BRST cohomology for SUSY invariant objects. We will also show that these solutions – up to some of them which turn out to be obviously trivial – are equivalent to solutions of an equivariant cohomology, defined in the WZ-gauge as a generalization of the $N = 1$ definition of Witten.

Section 2 presents an introduction to the superspace formalism, with superfields, superforms, supergauge invariance, superconnection, superghosts and BRST symmetry. The WZ-gauge fixing is recalled in Sect. 3, where also a generalized definition of equivariant cohomology is proposed. The problem of finding the observables and its solution are explained in Sect. 4. In Sect. 5, we write down the general result in the WZ-gauge, show that the integrands of the observables obey a set of generalized Witten's descent equations and that they are non-trivial in the sense of equivariant cohomology. Our conclusions are presented in Sect. 6. Superspace conventions and notation are given in Appendix A. Some useful propositions on the relative cohomologies of a general set of n coboundary operators, needed in the main text, are stated and proved in Appendix B.

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Appendix C contains the proof of another proposition of a more technical character. The WZ-gauge is recalled in Appendix D, where a one-to-one correspondence between the fields in this gauge and a set of covariant superfields is constructed.

This paper may be viewed as a continuation of both papers [14] and [10], the first reference dealing with the problem of the observables in the case $N = 1$, and the second one presenting an introduction to the superspace formalism for $N > 1$ and the reduction to the Wess–Zumino gauge. Preliminary results of the present work were presented in [11].

2 N-extended supersymmetry

“Shift supersymmetry” may describe the gauge fixing of gauge field configurations with null curvature, or alternatively with self-dual curvature. It appeared originally in the Donaldson–Witten model [1, 4], with one supersymmetry generator in four dimensional space-time. Generalizations of it for more than one supersymmetry generator and for any space-time dimension were described in [6, 10, 12, 14], where the superspace formalism has been developed. The purpose of this section is to review the formalism and fix the notation.

2.1 N superspace formalism

N supersymmetry is generated by the fermionic charges Q_I, $I = 1, \ldots, N$ obeying the Abelian superalgebra[1]

$$[Q_I, Q_J] = 0, \tag{2.1}$$

commuting with the space-time symmetry generators and the gauge group generators. The gauge group is some compact Lie group.

A representation of supersymmetry is provided by superspace, a supermanifold with D bosonic and N fermionic dimensions[2]. The respective coordinates are denoted by (x^μ, $\mu = 0, \ldots, D-1$), and (θ^I, $I = 1, \ldots, N$). A superfield is by definition a superspace function $F(x, \theta)$ which transforms as

$$Q_I F(x,\theta) = \partial_I F(x,\theta) \equiv \frac{\partial}{\partial\theta^I} F(x,\theta) \tag{2.2}$$

under an infinitesimal supersymmetry transformation.

An expansion in the coordinates θ^I of a generic superfield reads

$$F(x,\theta) = f(x) + \sum_{n=1}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n} f_{I_1\ldots I_n}(x), \tag{2.3}$$

where the space-time fields $f_{I_1\ldots I_n}(x)$ are completely antisymmetric in the indices $I_1 \ldots I_n$. We recall that all fields (and superfields) are Lie algebra valued. These fields and superfields may be generalized to p-forms and superfield p-forms:

$$\Omega_p(x,\theta) = \omega_p(x) + \sum_{n=1}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n}\omega_{p,\,I_1\ldots I_n}(x). \tag{2.4}$$

In (2.3) or (2.4), the components $n \geq 1$ are SUSY transforms of the lowest component. This may be viewed explicitly through the identity

$$\begin{aligned}\Omega_p(x,\theta) &= \exp\{\theta^I Q_I\}\,\omega_p(x) \\ &= \sum_{n=0}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n}\; Q_{I_n}\ldots Q_{I_1}\;\omega_p(x),\end{aligned} \tag{2.5}$$

which holds due to the easily checked superfield property $\partial_I \Omega_p = Q_I \Omega_p$ and the fact that a superfield is uniquely determined by its $\theta = 0$ component.

We shall also deal with superforms. A q-superform may be written as

$$\hat\Omega_q = \sum_{k=0}^{q} \Omega_{q-k;\,I_1\ldots I_k} d\theta^{I_1}\ldots d\theta^{I_k}, \tag{2.6}$$

where the coefficients $\Omega_{q-k;\,I_1\ldots I_k}$ are (Lie algebra valued) superfields which are space-time forms of degree $(q-k)$. They are completely symmetric in their indices since, the coordinates θ being anticommutative, the differentials $d\theta^I$ are commutative. The superspace exterior derivative is defined as

$$\hat d = d + d\theta^I \partial_I, \quad d = dx^\mu \partial_\mu, \tag{2.7}$$

and is nilpotent: $\hat d^{\,2} = 0$.

The basic superfield of the theory is the superconnection $\hat A$, a 1-superform:

$$\hat A = A + E_I d\theta^I, \tag{2.8}$$

with $A = A_\mu(x,\theta)dx^\mu$ a 1-form superfield and $E_I = E_I(x,\theta)$ a 0-form superfield. The superghost $C(x,\theta)$ is a 0-superform. We expand the components of the superconnection (2.8) as

$$A = a(x) + \sum_{n=1}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n} a_{I_1\ldots I_n}(x), \tag{2.9}$$

where the 1-form a is the gauge connection, and the 1-forms $a_{I_1\ldots I_n}$ its supersymmetric partners. The expansions of E_I and of the ghost superfield C read

$$\begin{aligned}E_I &= e_I(x) + \sum_{n=1}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n} e_{II_1\ldots I_n}(x), \\ C &= c(x) + \sum_{n=1}^{N} \frac{1}{n!}\theta^{I_1}\ldots\theta^{I_n} c_{I_1\ldots I_n}(x).\end{aligned} \tag{2.10}$$

[1] The bracket is here an anticommutator. Throughout this paper brackets will denote either commutators or anticommutators, according to the statistics of their arguments.

[2] Notation and conventions on superspace are given in Appendix A.

The infinitesimal supergauge transformations of the superconnection are expressed as the nilpotent BRST transformations

$$\mathcal{S}\hat{A} = -\hat{d}C - [C, \hat{A}], \quad \mathcal{S}C = -C^2, \quad \mathcal{S}^2 = 0. \tag{2.11}$$

In terms of component superfields we have

$$\begin{aligned} &\mathcal{S}A = -dC - [C, A], \quad \mathcal{S}E_I = -\partial_I C - [C, E_I], \\ &\mathcal{S}C = -C^2. \end{aligned} \tag{2.12}$$

The supercurvature

$$\hat{F} = \hat{d}\hat{A} + \hat{A}^2 = F_A + \Psi_I \, d\theta^I + \Phi_{IJ} \, d\theta^I d\theta^J \tag{2.13}$$

transforms covariantly:

$$\mathcal{S}\hat{F} = -[C, \hat{F}],$$

as well as its components

$$\begin{aligned} F_A &= dA + A^2, \quad \Psi_I = \partial_I A + D_A E_I, \\ \Phi_{IJ} &= \frac{1}{2}\left(\partial_I E_J + \partial_J E_I + [E_I, E_J]\right), \end{aligned} \tag{2.14}$$

where the covariant derivative with respect to the connection A is defined by $D_A(\cdot) = d(\cdot) + [A, (\cdot)]$

Special cases, $N = 1, 2$ and a discussion of the WZ-gauge can be found in [10].

3 Observables as equivariant cocycles

As discussed in [10] and recalled in Appendix D, it is possible to suppress all the supergauge degrees of freedom except the usual one corresponding to the $\theta = 0$ component c of the superghost C. This is the so-called "WZ-gauge fixing", obtained by fixing to zero a set of field components as in (D.1). In the WZ-gauge the supersymmetry generators must be modified into new operators $\tilde{Q}_I$, see (D.5)) differing from the previous ones, Q_I, by a field dependent gauge transformation. Accordingly, the algebra of the supercharges $\tilde{Q}_I$ closes up to field dependent gauge transformations as in (D.7).

A possible generalization for any N of Witten's equivariant cohomology [1] may be defined as follows, in the WZ-gauge.

(1) An "equivariant cocycle" is a *gauge invariant* local field polynomial – integrated or not – obeying the conditions

$$\tilde{Q}_I \Delta = 0, \quad I = 1, \ldots, N. \tag{3.1}$$

(2) A "trivial cocycle" is an equivariant cocycle of the form

$$\Delta = \tilde{Q}_1 \ldots \tilde{Q}_N \Delta', \tag{3.2}$$

where Δ' is *gauge invariant*.

(3) The "N-equivariant cohomology" is the set of equivalence classes of equivariant cocycles corresponding to the equivalence relation

$$\Delta_1 \approx \Delta_2 \quad \Leftrightarrow \quad \Delta_1 - \Delta_2 \ \text{ is trivial.} \tag{3.3}$$

This suggests the following generalization of Witten's definition.

Definition 1. An "observable" is an element of the N-equivariant cohomology.

Remark. In the superspace formalism, a SUSY invariant has necessarily the form of a total SUSY variation $\Delta = Q_1 \ldots Q_N \Delta'$, as stated in Corollary 1 of Proposition B.4[3].

4 Observables as supersymmetric BRST cocycles

4.1 Defining the problem

Computing the equivariant cohomology defined in the preceding section is presumably a difficult task. Instead of this, we shall generalize to arbitrary N the approach made in [14] for the case $N = 1$, defining observables, in the superspace formalism, as elements of the BRST cohomology in the space of the SUSY invariant space-time integrals of local field polynomials. Our task is thus to define and find global observables, of the form

$${}^K\Delta_{(d)} = \int_{M_d} {}^K\omega^0_d, \tag{4.1}$$

an integral of a p-form on a manifold M_d of dimension d. The labels of a form ${}^S\omega^g_p$ are defined as follows. S is a N component vector, with components $(s_1, s_2, \ldots, s_N)$ equal to the (non-negative) SUSY numbers, p is the (non-negative) form degree and g is the (non-negative) ghost-number.

The observable (4.1) has by definition to satisfy the BRST cocycle condition:

$$\mathcal{S}\ {}^K\Delta_d = 0, \qquad {}^K\Delta_d \neq \mathcal{S}\ {}^K\Delta'_d, \tag{4.2}$$

${}^K\Delta_{(d)}$ and ${}^K\Delta'_{(d)}$ being submitted to the SUSY constraints

$$Q_I\ {}^K\Delta_d = 0, \quad Q_I\ {}^K\Delta'_d = 0, \qquad I = 1, 2, \ldots, N. \tag{4.3}$$

4.2 General solution of the SUSY constraints

We shall solve (4.3) for ${}^K\Delta_d$, the solution for ${}^K\Delta'_d$ being analogous. From (4.3) we obtain the following equations for the integrand defined in (4.1):

$$\begin{aligned} &Q_I\ {}^{(k_1, \ldots k_I, \ldots k_N)}\omega^0_d + d\ {}^{(k_1, \ldots, k_I+1, \ldots k_N)}\omega^0_{d-1} = 0, \\ &I = 1, 2, \ldots, N. \end{aligned} \tag{4.4}$$

[3] The cohomology defined here should not be confused with that of the nilpotent operator $Q = \sum_I \epsilon^I Q_I$ or $\tilde{Q} = \sum_I \epsilon^I \tilde{Q}_I$, where the ϵ^I are commuting constant supersymmetry ghosts. The latter may be used in matters such as perturbative renormalization; see e.g. [16, 17] in the context of supersymmetric gauge theories.

Using Proposition B.4, we conclude that the general solution is given by

$$ {}^{K}\omega^0_d = Q_1 \dots Q_N \; {}^{K-E}\omega^0_d \; + \; d \; {}^{K}\varphi^0_{d-1}, \tag{4.5}$$

where E is the N dimensional vector $E = (1, 1, \dots, 1)$. From the identity (2.5) obeyed by any superfield and the fact that a product of more than N operators Q_I is identically vanishing, we see that we can replace the form ${}^{K-E}\omega^0_d$ in (4.5) by the superfield form

$$ {}^{K-E}\Omega^0_d(x,\theta) = \exp\{\theta^I Q_I\} \; {}^{K-E}\omega^0_d(x), \tag{4.6}$$

and thus we can write (4.5) as

$$ {}^{K}\omega^0_d(x) = Q_1 \dots Q_N \;\; {}^{K-E}\Omega^0_d(x,\theta) + d \; {}^{K}\varphi^0_{d-1}, \tag{4.7}$$

and write (4.1) as a superspace integral[4]:

$$ {}^{K}\Delta_d = \oint\!\!\!\!\!\!\int \; {}^{K-E}\Omega^0_d(x,\theta). \tag{4.8}$$

4.3 General solution of the BRST cocycle condition

The BRST invariance condition (4.2) yields, for the integrand of (4.1),

$$ \mathcal{S}\; {}^{H+E}\omega^0_d + d \; {}^{H+E}\omega^1_{d-1} = 0, $$

for some form ${}^{H+E}\omega^1_{d-1}$. From our previous result (4.7), this can be rewritten as

$$ \mathcal{S}\; Q_1 \dots Q_N \; {}^{H}\Omega^0_d + d \; {}^{H+E}\omega^1_{d-1} = 0. \tag{4.9}$$

For convenience we have redefined the SUSY numbers by putting $H = (h_1, \dots, h_N) = K - E = (k_1 - 1, \dots, k_N - 1)$. Let us show that the second term in the latter equation can also be written as a total SUSY variation. Applying Q_I to this equation we obtain

$$ dQ_I \; {}^{H+E}\omega^1_{d-1} = 0, \quad I = 1, \dots, N; $$

hence, due to the triviality of the cohomology of d in the space of local field functionals [18]:

$$ Q_I \; {}^{H+E}\omega^1_{d-1} = d(\dots), \quad I = 1, \dots, N. $$

Application of Proposition B.4 with $\delta_i = Q_I$ and $\delta_n = d$ then yields

$$ {}^{H+E}\omega^1_{d-1} = Q_1 \dots Q_N \;\; {}^{K-E}\Omega^1_{d-1}(x,\theta) + d(\dots), $$

and (4.9) takes the form

$$ Q_1 \dots Q_N \left(\mathcal{S} \; {}^{H}\Omega^0_d + d \; {}^{H}\Omega^1_{d-1} \right) = 0. \tag{4.10}$$

Now, application of Proposition B.2 gives

$$ \mathcal{S} \; {}^{H}\Omega^0_d + d \; {}^{H}\Omega^1_{d-1} + \sum_I Q_I \; {}^{H-E_I}\Omega^1_d = 0. \tag{4.11}$$

[4] See the definition given by (A.2) in Appendix A.

We shall show now that we can generate from the latter equation a complete set of "multi-descent equations" – a generalization of the notion of descent equations to the case of more than two antiderivative operators, which are here the operators $\mathcal{S}$, d, Q_1, …, Q_N. In order to do this, it is convenient to work with "truncated superforms" [14], a special case of the "truncated extended forms" introduced in Appendix C. We define a truncated superform of degree q by

$$ \check{\Omega}_q = \left[\hat{\Omega}_q \right]^{\,(\mathrm{tr})}, \tag{4.12}$$

where $\hat{\Omega}_q$ is a q-superform as defined by (2.6), which we may write as[5]

$$ \begin{aligned} \hat{\Omega}_q &= \sum_{k=0}^{q} \sum_{S}^{|S|=k} \; {}^{S}\Omega_{q-k}\, (d\theta)^S \\ &\equiv \sum_{k=0}^{q} \sum_{s_1,\dots,s_N}^{s_1+\dots+s_N=k} \; {}^{(s_1,\dots,s_N)}\Omega_{q-k}\, (d\theta_1)^{s_1} \dots (d\theta_N)^{s_N}, \end{aligned} \tag{4.13}$$

and where truncation, labeled by the exponent "${}^{(\mathrm{tr})}$", means discarding, in the expansion (4.13), all the terms of degree $q - k > d$, $s_I > h_I$ $(I = 1, \dots, I_N)$. Explicitly[6] we have

$$ \check{\Omega}_q = \sum_{k=\mathrm{Max}\{0,q-d\}}^{q} \; \sum_{S}^{S\le H,\,|S|=k} \; {}^{S}\Omega_{q-k}\, (d\theta)^S, \tag{4.14}$$

where $S \le H$ means $s_I \le h_I, \forall I$. We shall denote by $\check{\mathcal{E}}_{(d,H)}$ the space of truncated superforms defined by (4.12) and (4.14).

The exterior derivative $\check{d}$ acting in the space $\check{\mathcal{E}}_{(d,H)}$ is defined, according to (C.3), by

$$ \begin{aligned} \check{d}\,\check{\Omega}_q &= \left[\hat{d}\check{\Omega}_q \right]^{\,(\mathrm{tr})} \\ &= \sum_{k=\mathrm{Max}\{0,q+1-d\}}^{q} \; \sum_{S}^{S\le H,\,|S|=k} d \; {}^{S}\Omega_{q-k}\, (d\theta)^S \\ &\quad + \sum_{I=1}^{N} \left(\sum_{k=\mathrm{Max}\{0,q-d\}}^{q} \; \sum_{S}^{S\le H-E_I,\,|S|=k} \partial_I \; {}^{S}\Omega_{q-k} d\theta^I (d\theta)^S \right). \end{aligned} \tag{4.15}$$

As shown in Appendix C, $\check{d}$ is nilpotent. Proposition C.1 may be restated as follows.

Lemma 4.1 The cohomology of $\check{d}$ in the space $\check{\mathcal{E}}_{(d,H)}$ consists of the truncated superforms of maximal weights:

$$ \check{\Omega}_D = {}^{H}\; \Omega_d (d\theta)^H, \quad D \equiv d + |H|. \tag{4.16}$$

[5] The second summation in (4.12) is performed over all non-negative values of the SUSY numbers s_I, constrained by $|S| = k$, where $|S| \equiv \sum_{I=1}^N s_I$.

[6] Recall that all numbers such as form degree, SUSY number and ghost-number, are non-negative. As a general convention, any term which may appear with negative such numbers is understood to vanish.

One checks easily that (4.11) may be written in terms of truncated superforms:

$$\mathcal{S}\check{\Omega}^0_D + \check{d}\,\check{\Omega}^1_{D-1} = 0, \tag{4.17}$$

where $D = d+|H|$, and the upper index as usual denotes the ghost-number. The two truncated superforms appearing in this equation are

$$\check{\Omega}^0_D = {}^H\Omega^0_d, \tag{4.18}$$

$$\check{\Omega}^1_{D-1} = {}^H\Omega^1_{d-1}(d\theta)^H + \sum_{I=1}^{N} {}^{H-E_I}\,\Omega^1_d(d\theta)^{H-E_I}.$$

Applying $\check{d}$ to (4.17), we obtain

$$\check{d}\,\mathcal{S}\check{\Omega}^1_{D-1} = 0,$$

which, due to the triviality of the cohomology of $\check{d}$, solves in

$$\mathcal{S}\check{\Omega}^1_{D-1} + \check{d}\,\check{\Omega}^2_{D-2}.$$

Repeating the argument we finally obtain the $(\mathcal{S},\, \check{d}\,)$ descent equations

$$\mathcal{S}\check{\Omega}^g_{D-g} + \check{d}\,\check{\Omega}^{g+1}_{D-g-1}, \quad g = 0,\ldots,D. \tag{4.19}$$

These are the "multi-descent equations" which characterize the observable (4.1), of dimension d and SUSY weight $H+E$. These equations read explicitly[7]

$$\mathcal{S}\,{}^S\Omega^g_p + d\;{}^S\Omega^{g+1}_{p-1} + \sum_{I=1}^{N} Q_I\;{}^{S-E_I}\;\Omega^{g+1}_p = 0, \tag{4.20}$$

$$g = D-p-|S|, \quad s_I = 0,\ldots,h_I, \quad p = 0,\ldots,d,$$

where $S = (s_1,\ldots,s_N)$, $H = (h_1,\ldots,h_N)$, $|S| = \sum_1^N s_I$, $|H| = \sum_1^N h_I$ and $D = d+|H|$.

4.4 Solving the multi-descent equations

4.4.1 Cohomology of $\mathcal{S}$

With the purpose of resolving the multi-descent equations (4.19), our first task will be to resolve the cohomology of the BRST operator $\mathcal{S}$ in the space $\mathcal{E}_S$ of the local polynomials in the superfield forms of the theory and of their derivatives, and then in the space $\check{\mathcal{E}}_{(d,H)}$ of the truncated superforms (4.12). An obvious algebraic basis of $\mathcal{E}_S$ is given by the set of superfields (2.9)–(2.10) and their derivatives:

$$\{A,\, E_I,\, C,\, A_{,\,I_1\ldots I_n},\, E_{I,\,I_1\ldots I_n},\, C_{,\,I_1\ldots I_n},\, dA,\, dE_I,\, dC,$$
$$dA_{,\,I_1\ldots I_n}, dE_{I,\,I_1\ldots I_n},\, dC_{,\,I_1\ldots I_n}\,;\, n\geq 1\}\,, \tag{4.21}$$

where we use the notation $X_{,\,I_1\ldots I_n} = \partial_{I_1}\ldots\partial_{I_n}X$ for the θ-derivatives.

[7] For the case of one SUSY generator [14] they are called bi-descent equations.

A more convenient basis is one that consists of BRST doublet superfields and of covariant superfields. The BRST doublets are identified as

$$\left(E^{(\mathrm{A})}_{I_1\ldots I_n},\, K^{(\mathrm{A})}_{I_1\ldots I_n}\right), \quad n\geq 1, \tag{4.22}$$

where the $E^{(\mathrm{A})}$ are the completely antisymmetrized θ-derivatives of E_I, and the $K^{(\mathrm{A})}$ are their BRST variations:

$$E^{(\mathrm{A})}_{I_1\ldots I_n} = E_{[I_1,\,I_2\ldots I_n]}, \quad K^{(\mathrm{A})}_{I_1\ldots I_n} = \mathcal{S}E^{(\mathrm{A})}_{I_1\ldots I_n},$$
$$\mathcal{S}K^{(\mathrm{A})}_{I_1\ldots I_n} = 0.$$

The remainder of the basis is given by the set of covariant superfields constructed in Appendix D and shown in (D.8). A complete algebraic basis is thus provided by

$$\left\{E^{(\mathrm{A})}_{I_1\ldots I_n},\, K^{(\mathrm{A})}_{I_1\ldots I_n},\, dE^{(\mathrm{A})}_{I_1\ldots I_n},\, dK^{(\mathrm{A})}_{I_1\ldots I_n}\,;\, n\geq 1\right\}$$
$$\bigoplus\left\{A,\, C,\, F_A,\, dC,\, \Psi^{(\mathrm{A})}_{I_1\ldots I_n}\,(n\geq 1),\, \Phi^{(M)}_{I_1\ldots I_n}\,(n\geq 2),\right.$$
$$\left. D_A\Psi^{(\mathrm{A})}_{I_1\ldots I_n}\,(n\geq 1),\, D_A\Phi^{(M)}_{I_1\ldots I_n}\,(n\geq 2)\right\}. \tag{4.23}$$

A first obvious conclusion is that the fields $E^{(\mathrm{A})}_{I_1\ldots I_n}$ and $K^{(\mathrm{A})}_{I_1\ldots I_n}$ and their derivatives do not contribute to the cohomology of $\mathcal{S}$, since they are BRST doublets [17]. Moreover, the fields $\Psi^{(\mathrm{A})}_{I_1\ldots I_n}$ and $\Phi^{(M)}_{I_1\ldots I_n}$ can be viewed as "matter" fields, transforming in the adjoint representation of the gauge group. Then, as a consequence of the general results of [18], we can conclude that the cohomology of $\mathcal{S}$ in the space $\mathcal{E}_S$ consists of the local polynomials generated by the cocycles

$$\theta_r(C) \quad (r = 1,\ldots,\text{rank } G) \tag{4.24}$$

and

$$P^{\mathrm{inv}}(F,\, \Psi^{(\mathrm{A})}_{I_1\ldots I_n},\, \Phi^{(M)}_{I_1\ldots I_n},\, D_A\Psi^{(\mathrm{A})}_{I_1\ldots I_n},\, D_A\Phi^{(M)}_{I_1\ldots I_n}), \tag{4.25}$$

where $P^{\mathrm{inv}}(\ldots)$ is any gauge invariant polynomial of its arguments, and where θ_r is the ghost cocycle associated in a standard way [18] to the rth Casimir operator of the gauge group G and given, as a function of the superghost C, by

$$\theta_r(C) = (-1)^{m_r-1}\,\frac{m_r!\,(m_r-1)!}{g_r!}\,\mathrm{Tr}\,C^{g_r}$$
$$(\,g_r = 2m_r-1\,,\ r = 1,\ldots,\text{rank } G\,), \tag{4.26}$$

where the index r labels the rth Casimir operator of the structure group (gauge group) G, whose degree is denoted by m_r. An obvious generalization of the results of [18] shows that the cocycles (4.26) are related by superdescent equations involving superforms $\left[\hat{\theta}_r\right]^{g_r-p}_p$ of form degree $p\geq 0$ and ghost-number g_r-p:

$$\mathcal{S}\left[\hat{\theta}_r\right]^{g_r-p}_p + \hat{d}\left[\hat{\theta}_r\right]^{g_r-p+1}_{p-1} = 0 \quad (\,p = 0,\ldots,g_r\,), \tag{4.27}$$

with $\left[\hat{\theta}_r\right]_0^{g_r} = \theta_r(C)$ and $\hat{d}\left[\hat{\theta}_r\right]_{g_r}^0 = f_r(\hat{F})$, where

$$f_r(\hat{F}) = \operatorname{Tr} \hat{F}^{m_r} \quad (r = 1, \ldots, \operatorname{rank} G), \tag{4.28}$$

$\hat{F} = \hat{d}\hat{A} + \hat{A}^2$ being the supercurvature (2.13). According to the last one of (4.27), the "bottom" superform $\left[\hat{\theta}_r\right]_{g_r}^0$ is the Chern–Simons superform of degree g_r associated to the rth Casimir operator.

A straightforward generalization of the result (4.24)–(4.25) from superfield forms to truncated superforms yields the following lemma.

Lemma 4.2 The cohomology of $\mathcal{S}$ in the functional space $\check{\mathcal{E}}_{(d,H)}$ is given by the truncated forms whose non-vanishing coefficients are polynomials in the superfield forms given in (4.25).

4.4.2 Cohomology of $\mathcal{S}$ modulo $\check{d}$

The resolution of the cohomology of $\mathcal{S}$ modulo $\check{d}$ in the space $\check{\mathcal{E}}_{(d,H)}$ of truncated superforms, i.e. the resolution of the multi-descent equations (4.19) has been done in Appendix A.4 of [14] for the case $N = 1$. The computation, relying on the cohomologies of $\check{d}$ and $\mathcal{S}$ in $\check{\mathcal{E}}_{(d,H)}$ (Lemmas 4.1 and 4.2), applies as well to arbitrary N, thus leading to the following proposition.

Proposition 4.3 The general solution of the multi-descent equations (4.19) corresponding to the observable (4.1) is generated, at ghost-number zero, by two classes of solutions. The first one is given by the superfield forms (recall that $H = K - E$) Solution of Type I:

$${}^{H}\Omega_d^0\,(d\theta)^H = \left[\left[\hat{\theta}_{r_1}\right]_{g_{r_1}}^0 f_{r_2}(\hat{F}) \ldots f_{r_L}(\hat{F})\right]_{S=H,\,p=d}$$

with

$$|H| + d = D, \quad D = 2\sum_{i=1}^{L} m_{r_i} - 1, \quad L \geq 1, \tag{4.29}$$

where the Chern–Simons superform $\left[\hat{\theta}_r\right]_{g_r}^0$ and supercurvature invariant $f_r(\hat{F})$ are defined by (4.26)–(4.28).

The second class of solutions depends on the superfield forms F, Ψ and $\Phi^{(M)}$ appearing in the cohomology of $\mathcal{S}$ (see (4.25)) and it is given by Solution of Type II:

$${}^{H}\Omega_d^0 = {}^{H}\mathcal{Z}_d^0\left(F, \Psi_{I_1\ldots I_n}^{(\mathrm{A})}, \Phi_{I_1\ldots I_n}^{(M)}, D_A\Psi_{I_1\ldots I_n}^{(\mathrm{A})}, D_A\Phi_{I_1\ldots I_n}^{(M)}\right), \tag{4.30}$$

Here, ${}^{H}\mathcal{Z}_d^0$ is an arbitrary invariant polynomial of its arguments, which has a form degree d and SUSY-numbers given by H, and which is non-trivial in the sense that

$${}^{H}\mathcal{Z}_d^0 \neq d\,{}^{H}\Phi_{d-1}^0 + \sum_{I=1}^{N} Q_I\,{}^{H-E_I}\Phi_d^0.$$

Let us make two remarks.
(1) As in the $N = 1$ case [14] the superspace integral (see (4.8)) of any solution of the type (4.30):

$${}^{K}\Delta_d = \oint {}^{K-E}\mathcal{Z}_d^0,$$

which belongs to the BRST cohomology in the space of the SUSY invariant BRST cocycles, is in fact trivial from the point of view of the equivariant cohomology defined in Sect. 3. Indeed, being the space-time integral of a superfield form

$${}^{K}\omega_d^0 = -\frac{1}{N!} Q_1 \ldots Q_N\,{}^{K-E}\mathcal{Z}_d^0,$$

where ${}^{K-E}\mathcal{Z}_d^0$ is gauge invariant, it reduces in the WZ-gauge to an equivariantly trivial expression:

$$\int_{M_d} {}^{K}\omega_d^0 = -\frac{1}{N!}\tilde{Q}_1 \ldots \tilde{Q}_N \int_{M_d} {}^{K-E}z_d^0,$$

where ${}^{K-E}z_d^0$ is gauge invariant.

This follows from the fact that the operators Q_I and $\tilde{Q}_I$ coincide, when applied to gauge invariant expressions, as seen from (D.5).
(2) As we shall see in Sect. 5, solutions of the type (4.29) are not trivial from the point of view of the equivariant cohomology. In the case $N = 1$ [14] they are the Witten–Donaldson observables [1]. The cases $N > 1$ thus offer a generalization of the latter ones.

4.5 Superform expression of the observables

Let us consider the untruncated version of (4.19) involving full superforms (see (2.6)):

$$\begin{aligned} &\mathcal{S}\hat{\Omega}_{D-g}^{g} + \hat{d}\hat{\Omega}_{D-g-1}^{g+1}, \\ &g = 0, \ldots, D \\ &(D = |H| + d). \end{aligned} \tag{4.31}$$

It can be shown [14] on the basis of the results of [18] concerning BRST cohomology that the general solution of the superdescent equations (4.31), which contains superforms down to and including ghost-number $g = 0$, is given by

$$\begin{aligned} \hat{\Omega}_{D-g_{r_1}+p}^{g_{r_1}-p} &= \left[\hat{\theta}_{r_1}\right]_p^{g_{r_1}-p} f_{r_2}(\hat{F}) \ldots f_{r_L}(\hat{F}), \\ &p = 0, \ldots, g_{r_1}, \end{aligned} \tag{4.32}$$

with the $\left[\hat{\theta}_{r_1}\right]_p^{g_{r_1}-p}$ and the supercurvature invariant $f_r(\hat{F})$ defined by (4.26)–(4.28). The superfield components ${}^{S}\Omega_p^g$, with $|S| + g + p = D$, of these superforms are clearly solutions of the multi-descent equations (4.19) since the latter is a subsystem of (4.31). The corresponding observables are given by the superspace integrals of the superfield components of the expansion

$$\hat{\Omega}^0_D = \left[\hat{\theta}_{r_1}\right]^0_{g_{r_1}} f_{r_2}(\hat{F})\dots f_{r_L}(\hat{F}) \equiv \sum_H^{|H|\le D} {}^H\Omega^0_{D-|H|}\,(d\theta)^H, \tag{4.33}$$

i.e. (see (4.1) and (4.8))

$$\begin{aligned} {}^K\Delta_d &= \oint {}^H\Omega^0_d, \\ d &= D-|H|, \\ K &= H+(1,\dots,1). \end{aligned} \tag{4.34}$$

On the other hand, we see from (4.33) taken with all possible values of D and of the numbers g_{r_i} that the components of the solutions of the superdescent equations (4.31) span all the solutions of type I (4.29) of the multi-descent equations (4.19). Thus we have the following.

Proposition 4.4 If $\hat{\Omega}^0_D$ represents the general solution of the superdescent equations (4.31), then the superfield forms ${}^H\Omega^0_{D-|H|}$ defined by the expansion of $\hat{\Omega}^0_D$ in (4.33) represent the general solution of type (4.29) of the multi-descent equations (4.19).

We note for the sake of completeness that the superforms $\hat{\Omega}^{g_{r_1}-p}_{D-g_{r_1}+p}$ obey the system of superdescent equations

$$\mathcal{S}\hat{\Omega}^{g_{r_1}-p}_{D-g_{r_1}+p} + \hat{d}\hat{\Omega}^{g_{r_1}-p+1}_{D-g_{r_1}+p-1}, \quad p=0,\dots,g_{r_1},$$

involving ghost-numbers up to the value g_{r_1}, which is less than the maximum possible value D if $L>2$.

A convenient way of representing the observables (4.34) and deducing interesting properties of them, is based on the identity $\hat{d}\left[\hat{\theta}_{r_1}\right]^0_{g_{r_1}} = f_{r_1}(\hat{F})$ for the Chern–Simons form, and the expansion

$$\begin{aligned} \hat{d}\hat{\Omega}^0_D &= f_{r_1}(\hat{F})\dots f_{r_L}(\hat{F}) \\ &= f_{r_1}(F)\dots f_{r_L}(F) + \sum_S^{1\le|S|\le D+1} {}^SW^0_{D+1-|S|}(d\theta)^S, \end{aligned} \tag{4.35}$$

with the first term being a d-derivative, and

$${}^SW^0_{D+1-|S|} = \sum_{I=1}^N Q_I\ {}^{S-E_I}\Omega^0_{D+1-|S|} + d\ {}^S\Omega^0_{D-|S|}. \tag{4.36}$$

Integrating both sides of the latter equation in superspace, we see that we can write (4.34) as

$$\begin{aligned} {}^K\Delta_d &= \oint {}^H\Omega^0_d \\ &= (-1)^{N-J}\int_{M_d} Q_1\dots\widehat{Q_J}\dots Q_N\ {}^{H+E_J}W^0_d, \end{aligned} \tag{4.37}$$

with $H=K-E$, $d=D-|H|=D+N-K$, where the notation $\widehat{X}$ means suppression of the factor X. The value of J in the right-hand side is arbitrary. Let us show that the expression is in fact independent of J as it should. Applying the nilpotent operator $\hat{d}$ on (4.35) we obtain the descent equations

$$\begin{aligned} &\sum_{I=1}^N Q_I\ {}^{H-E_I}W^0_{D+2-|H|} + d\ {}^HW^0_{D+1-|H|} = 0, \\ &\qquad 1\le|H|\le D+1, \\ &\sum_{I=1}^N Q_I\ {}^{H-E_I}W^0_0 = 0, \quad |H| = D+2. \end{aligned} \tag{4.38}$$

Considering the difference of the expressions (4.37) obtained for two values of J, which we may choose without loss of generality as $J=N$ and $N-1$, respectively, we obtain

$$\begin{aligned} &\int_{M_d} Q_1\dots Q_{N-2}\left(Q_{N-1}\ {}^{H'-E_{N-1}}W^0_d + Q_N\ {}^{H'-E_N}W^0_d\right), \\ &H' = H+E_N+E_{N-1}, \end{aligned}$$

which, by virtue of (4.38) for $|H|=D+2-d$, reads

$$-\int_{M_d} Q_1\dots Q_{N-2}\sum_{I=1}^{N-2} Q_I\ {}^{H'-E_I}W^0_d = 0,$$

and which vanishes due to the nilpotency of the operators Q_I.

5 Witten's observables and descent equations

Let us rewrite the integral (4.37), expressing a generic observable, as a space-time integral:

$${}^K\Delta_d = \int_{M_d} {}^K\omega_d, \tag{5.1}$$

the integrand being defined up to a total space-time derivative. Let us define the latter as

$${}^K\omega_d = \sum_{J=1}^N (-1)^{N-J}\alpha_J\,Q_1\dots\widehat{Q_J}\dots Q_N\ {}^{H+E_J}W^0_d\Big|_{\theta=0}, \tag{5.2}$$

with $\sum_J \alpha_J = 1$. It is clear from the discussion at the end of the last subsection that, to the contrary of its integrand, the integral does not depend on the arbitrary numbers α. Calculating

$$Q_I\ {}^K\omega_d = (-1)^{N-1}\alpha_I\,Q_1\dots Q_N\ {}^{H+E_I}W^0_d$$

and

$$
\begin{aligned}
&d^{\,K+E_I}\omega_{d-1} \\
&= \sum_{J=1}^{N}(-1)^{J-1}\alpha_J\, Q_1\dots\widehat{Q_J}\dots Q_N\, d^{\,H+E_J+E_I}W^0_{d-1}\Big|_{\theta=0} \\
&= (-1)^N\alpha_I\, Q_1\dots Q_N\,{}^{H+E_I}W^0_d ,
\end{aligned}
$$

where we have used (4.38) for the last equality, we conclude that the integrands (5.2) obey the descent equations

$$
\begin{aligned}
&Q_I\,{}^K\omega_d + \alpha_I\, d^{\,K+E_I}\omega_{d-1} = 0, \\
&\qquad I = 1,\dots,N, \quad N \le |K| \le D-d+N. \qquad (5.3)
\end{aligned}
$$

We can now go to the WZ-gauge (see Appendix D). The forms ${}^K\omega_d$ being gauge invariant functions of the covariant superfields (D.8) taken at $\theta = 0$, they reduce to correspondent gauge invariant functions of the covariant WZ-gauge fields (D.2) by virtue of the correspondence (D.10). Moreover, since the expressions are gauge invariant, the applications of the generators $\tilde{Q}_I$ and Q_I are identical. Hence, (5.3) reduce to

$$
\begin{aligned}
&\tilde{Q}_I\,{}^K\omega_d + \alpha_I\, d^{\,K+E_I}\omega_{d-1} = 0, \\
&\qquad I = 1,\dots,N, \quad N \le |K| \le D-d+N, \qquad (5.4)
\end{aligned}
$$

which are the possible generalizations to arbitrary N of Witten's descent equations [1].

Specializing to two particular values of the set of numbers α_I, we would obtain

$$
\begin{aligned}
&\tilde{Q}_I\,{}^K\omega_d + \frac{1}{N}\, d^{\,K+E_I}\omega_{d-1} = 0 \qquad \left(\alpha_I = \frac{1}{N}\right), \\
&\left.\begin{aligned}&\tilde{Q}_J\,{}^K\omega_d + d^{\,K+E_J}\omega_{d-1} = 0\\ &\tilde{Q}_I\,{}^K\omega_d = 0, \quad I \ne J\end{aligned}\right\} \\
&\qquad\qquad (\alpha_J = 1\,,\ \alpha_I = 0\,,\ I \ne J). \qquad (5.5)
\end{aligned}
$$

Let us recall that in all systems of equations above, any term with negative SUSY number or form degree is assumed to vanish.

Equations (5.4) show that our solutions, which solve Witten's descent equations, are indeed Witten's observables, their space-time integrals being $\tilde{Q}_I$-invariant for any I. It remains to show that they are non-trivial in the sense defined in Sect. 3. For this it is sufficient to check the non-triviality of the w's of highest SUSY numbers (hence of zero form degree). The latters are $\tilde{Q}_I$-invariant, and read (see e.g. (5.2) for $\alpha_1 = 1$, $\alpha_I = 0\,, I \ne 1$, and (4.35))

$$
\begin{aligned}
&{}^{I_1J_1\dots I_nJ_n}w_0 = (-1)^{N-1}Q_2\dots Q_N \\
&\quad\times \left(\Phi_{I_1J_1},\dots,\Phi_{I_nJ_n}\right)\big|_{\text{symmetrized in }(I_1,\dots,J_n)} ,
\end{aligned}
$$

with $n = \sum_{r=1}^{L} m_r$, where we use the notation (4.13) and $(X_1,\dots,X_n)$ is a symmetric invariant polynomial of its arguments. In the WZ-gauge

$$
\begin{aligned}
&{}^{I_1J_1\dots I_nJ_n}w_0 = (-1)^{N-1}\tilde{Q}_2\dots \tilde{Q}_N \\
&\quad\times \left(e^{\mathrm{M}}_{I_1J_1},\dots,e^{\mathrm{M}}_{I_nJ_n}\right)\big|_{\text{symmetrized in }(I_1,\dots,J_n)} ,
\end{aligned}
$$

with $n = \sum_{r=1}^{L} m_r$.

It is easy to check that the e^{M}_{IJ} can never be written as a $\tilde{Q}_1$-variation. Hence w_0 cannot be written as a full $\tilde{Q}_1\tilde{Q}_2\dots\tilde{Q}_N$-variation and thus belongs to the equivariant cohomology.

6 Conclusion and open problems

The results on the classification of the observables known for the topological Yang–Mills theories with one supersymmetry generator were generalized to the case of theories defined with more supersymmetry generators. We have obtained a complete classification of the observables according to their definition as non-trivial BRST cocycles in the space of supersymmetry invariant local functionals.

Although our solutions are solutions of the equivariant cohomology problem defined in Sect. 3, we have no proof that it provides the complete solution of the latter. However, we have also found generalized Witten's descent equations for the integrands of the observables and showed their non-triviality in the equivariant cohomology sense.

Our results are formal, being established in the classical approximation. Their interpretation at the quantum level as topological invariants remains an open problem in the general case of arbitrary numbers of SUSY generators and space-time dimensions, although some results are known for special cases, in particular for $N = 2$ [6,9].

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Appendix A: N-supersymmetry and superspace

(D,N)-superspace bosonic coordinates are denoted by x^μ, $\mu = 0,\dots,D-1$, the fermionic (Grassmann, or anticommuting) coordinates being denoted by θ^I, $I - 1,\dots,N$. The N supersymmetry generators Q_I are represented on superfields $F(x,\theta)$ by

$$
Q_I F = \partial_I F \equiv \frac{\partial}{\partial\theta^I}F,
$$

where, by definition, $\partial_K\theta^J = \delta^J_K$. Further conventions and properties concerning the θ-coordinates are the following:

$$
\begin{aligned}
\theta^N &= \epsilon_{I_1\dots I_N}\theta^{I_1}\dots\theta^{I_N} = N!\,\theta^1\dots\theta^N, \\
(\partial_\theta)^N &= \epsilon^{I_1\dots I_N}\partial_{I_1}\dots\partial_{I_N} = N!\,\partial_1\dots\partial_N, \qquad (A.1)\\
(\partial_\theta)^N\theta^N &= -(N!)^2,
\end{aligned}
$$

where $\epsilon^{I_1\dots I_N}$ is the completely antisymmetric tensor of rank N, with the conventions

$$
\epsilon^{1\dots N} = 1, \quad \epsilon_{I_1\dots I_N} = (-1)^{N+1}\,\epsilon^{I_1\dots I_N}.
$$

One may define the conserved supersymmetry number – the *SUSY number* – attributing the value 1 to the generators Q_I, hence -1 to the θ-cordinates. The SUSY number of each field component is then deduced from the SUSY number given to each superfield.

Superspace integration of a superfield form $\Omega_p(x,\theta)$ is defined by the integrals

$$\oint \Omega_p(x,\theta) = \int_{M_p} \int d^N\theta\ \Omega_p(x,\theta), \tag{A.2}$$

where the x-space integral is made on some p dimensional (sub)manifold M_p, and the θ-space integral is the Berezin integral defined by

$$\int d^N\theta\ \ldots = -\frac{1}{(N!)^2}(\partial_\theta)^N \ldots,$$

such that $\int d^N\theta\ \theta^N = 1$.

Appendix B: Some useful propositions

The propositions and proofs presented here are generalizations of results given in [18]. They hold for both usual forms and superfield forms.

Definitions and notation. Let $\omega^{(s_1,\ldots,s_n)}$ be forms whose weights s_i are associated to n operators δ_i, $(i=1,\ldots,n)$, nilpotent and anticommuting, i.e. $\{\delta_i,\delta_j\}=0$. The cohomology group of each operator δ_i is trivial by hypothesis. If some of the weights s_i are negative we have, by convention, $\omega^{(s_1,\ldots,s_n)}=0$. We shall use the condensed notation

$$\begin{aligned}
&\omega^{(s_1,\ldots,s_n)} = \omega^S, \quad S=(s_1,\ldots,s_n), \quad |S| = \sum_{i=1}^n s_i,\\
&E_i = (0,\ldots,1,\ldots,0)\\
&\quad\text{(unique non-vanishing component is a 1}\\
&\qquad\text{at the } i\text{th position)},\\
&E = \sum_{i=1}^n E_i = (1,1,\ldots,1),\\
&S-T = (s_1-t_1,\ldots,s_n-t_n)\\
&S \le T \ \Leftrightarrow\ s_i \le t_i\,,\ i=1,\ldots n.
\end{aligned} \tag{B.1}$$

The forms ω^S may be fields or superfields. We have

Proposition B.1 Let the set of forms $\{\omega^{T-E_i}\,|\,i=1,\ldots,n\} = \{\omega^{(t_1,\ldots,t_i-1,\ldots,t_n)}\,|\,i=1,\ldots,n\}$ satisfy the cocycle condition

$$\sum_{i=1}^n \delta_i\ \omega^{T-E_i} = 0. \tag{B.2}$$

(1) The set $\{\omega^{T-E_i}\,|\,i=1,\ldots,n\}$ can be extended to an *extended form* $\tilde\omega$, defined by

$$\tilde\omega = \sum_S^{|S|=|T|-1} \omega^S, \tag{B.3}$$

such that

$$\tilde\delta\ \tilde\omega = 0\ ; \qquad \tilde\delta = \sum_{i=1}^n \delta_i\,. \tag{B.4}$$

(2) There exists an extended form

$$\tilde\varphi = \sum_S^{|S|=|T|-2} \varphi^S, \tag{B.5}$$

where $\tilde\omega$ and $\tilde\varphi$ satisfy

$$\tilde\omega = \tilde\delta\ \tilde\varphi. \tag{B.6}$$

Corollary 1 The cohomology of $\tilde\delta$ is trivial.

Corollary 2 The general solution of (B.2) for any ω^{T-E_i} is given by

$$\omega^{T-E_i} = \sum_{j=1}^n \delta_j \varphi^{T-E_i-E_j}, \quad j=1,\ldots,n, \tag{B.7}$$

where all $\varphi^{T-E_i-E_j}$ for i, $j=1,\ldots,n$ are components of a single extended form $\tilde\varphi$.

To proof Proposition B.1, we note the following. We will proceed by induction from the case $n=2$ which will be first treated explicitly.
Case $n=2$. In this case, (B.2) is given by

$$\delta_1\omega^{(t_1-1,t_2)} + \delta_2\omega^{(t_1,t_2-1)} = 0. \tag{B.8}$$

The proof of Part 1 is as follows. Applying δ_1 to (B.8) we obtain $\delta_2\delta_1\omega^{(t_1,t_2-1)}=0$. Remembering that the cohomology of δ_2 is trivial, we deduce the existence of a form $\omega^{(t_1+1,t_2-2)}$, such that

$$\delta_1\omega^{(t_1,t_2-1)} + \delta_2\omega^{(t_1+1,t_2-2)} = 0. \tag{B.9}$$

Repeating successively this procedure we finally get

$$\delta_1\omega^{(|T|-1,0)} = 0, \quad |T| = t_1+t_2. \tag{B.10}$$

We have thus obtained the set of equations

$$\delta_1\omega^{(t_1+k-1,t_2-k)} + \delta_2\omega^{(t_1+k,t_2-k-1)} = 0, \quad 0\le k\le t_2.$$

Applying now δ_2 to (B.8) and using the triviality of the cohomology of δ_1, we obtain in an analogous way

$$\delta_1\omega^{(t_1-k'-1,t_2+k')} + \delta_2\omega^{(t_1-k',t_2+k'-1)} = 0, \quad 0\le k'\le t_1.$$

These last two systems of equations can be put into a unique set:

$$\begin{aligned}
&\delta_1\omega^{(t_1+p-1,t_2-p)} + \delta_2\omega^{(t_1+p,t_2-p-1)} = 0,\\
&-t_1 \le p \le t_2,
\end{aligned} \tag{B.11}$$

which is exactly (B.4) written in components, corresponding to the extended form and to the extended operator

$$\tilde\omega = \sum_{p=-t_1}^{t_2} \omega^{(t_1+p-1,t_2-p)}, \quad \tilde\delta = \delta_1+\delta_2. \tag{B.12}$$

The proof of Part 2 is as follows. Equation (B.11) for $p = -t_1$ is (B.10). From the triviality of the cohomology of δ_2 we obtain the general solution

$$\omega^{(0,|T|-1)} = \delta_2 \varphi^{(0,|T|-2)}, \tag{B.13}$$

and by substituting (B.13) in (B.11) for $p = -t_1 + 1$, we obtain

$$\delta_2 \left[-\delta_1 \varphi^{(0,|T|-2)} + \omega^{(1,|T|-2)} \right] = 0, \tag{B.14}$$

whose general solution for $\omega^{(1,|T|-2)}$ is

$$\omega^{(1,|T|-2)} = \delta_1 \varphi^{(0,|T|-2)} + \delta_2 \varphi^{(1,|T|-3)}. \tag{B.15}$$

The procedure continues until (B.11) for $p = t_2 - 1$, leading finally to

$$\omega^{(t_1+p-1,t_2-p)} = \delta_1 \varphi^{(t_1+p-2,t_2-p)} + \delta_2 \varphi^{(t_1+p-1,t_2-p-1)}, \quad -t_1 + 1 \le p \le t_2, \tag{B.16}$$

with the last equation, for $p = t_2$, being identically satisfied. Notice that the set (B.16) can be put in the form (B.6) with

$$\tilde{\varphi} = \sum_{p=-t_1+2}^{t_2} \varphi^{(t_1+p-2,\, t_2+p)}. \tag{B.17}$$

The general case

In order to establish the proof for general n we suppose that the proposition is valid for $(n-1)$. Applying for example δ_1 on (B.2), we get

$$\sum_{i=2}^{n} \delta_i \delta_1 \omega^{(t_1,t_2,\ldots,t_i-1,\ldots,t_n)} = 0. \tag{B.18}$$

By the induction hypothesis there exist $(n-1)$ forms $\omega^{(t_1+1,t_2-2,t_3,\ldots,t_n)}$ and $\omega^{(t_1+1,t_2-1,t_3,\ldots,t_i-1,\ldots,t_n)}$, with $i = 3, \ldots, n$, such that

$$\delta_1 \omega^{(t_1,t_2-1,t_3,\ldots,t_n)} + \delta_2 \omega^{(t_1+1,t_2-2,t_3,\ldots,t_n)} + \sum_{i=3}^{n} \delta_i \omega^{(t_1+1,t_2-1,t_3,\ldots,t_i-1,\ldots,t_n)} = 0. \tag{B.19}$$

Repeating the procedure we get

$$\delta_1 \omega^{(t_1+k-1,t_2-k,t_3,\ldots,t_n)} + \delta_2 \omega^{(t_1+k,t_2-k-1,t_3,\ldots,t_n)} + \sum_{i=3}^{n} \delta_i \omega^{(t_1+k,t_2-k,t_3,\ldots,t_i-1,\ldots,t_n)} = 0, \tag{B.20}$$

with $0 \le k \le t_2$. Beginning the same procedure from (B.2), but applying δ_2 and considering the cohomology of $\delta_1, \delta_3, \ldots, \delta_n$, we obtain

$$\delta_1 \omega^{(t_1-k'-1,t_2+k',t_3,\ldots,t_n)} + \delta_2 \omega^{(t_1-k',t_2+k'-1,t_3,\ldots,t_n)} + \sum_{i=3}^{n} \delta_i \omega^{(t_1-k',t_2+k',t_3,\ldots,t_i-1,\ldots,t_n)} = 0, \quad 0 \le k' \le t_1. \tag{B.21}$$

We can unify the last two set of equations in the following one:

$$\delta_1 \omega^{(t_1+p-1,t_2-p,t_3,\ldots,t_n)} + \delta_2 \omega^{(t_1+p,t_2-p-1,t_3,\ldots,t_n)} + \sum_{i=3}^{n} \delta_i \omega^{(t_1+p,t_2-p,t_3,\ldots,t_i-1,\ldots,t_n)} = 0, \quad -t_1 \le p \le t_2. \tag{B.22}$$

Introducing now the 2-extended operator $\tilde{\delta}_{(1,2)} = \delta_1 + \delta_2$ and the sets of 2-extended forms

$$\tilde{\omega}_{(1,2)}^{(\tilde{t},t_3,\ldots,t_i-1,\ldots,t_n)} = \sum_{s_1,s_2}^{s_1+s_2=\tilde{t}} \omega^{(s_1,s_2,t_3,\ldots,t_i-1,\ldots t_n)}, \quad i = 3, \ldots, n,$$
$$\tilde{\omega}_{(1,2)}^{(\tilde{h}-1,t_3,\ldots,t_n)} = \sum_{s_1,s_2}^{s_1+s_2=\tilde{t}-1} \omega^{(s_1,s_2,t_3,\ldots t_n)}, \tag{B.23}$$

we can rewrite (B.22) as

$$\tilde{\delta}_{(1,2)} \tilde{\omega}_{(1,2)}^{(\tilde{t}-1,t_3,\ldots,t_n)} + \sum_{i=3}^{n} \delta_i \tilde{\omega}_{(1,2)}^{(\tilde{t},t_3,\ldots,t_i-1,\ldots,t_n)} = 0, \quad \tilde{t} = t_1 + t_2. \tag{B.24}$$

Observe that the operator $\tilde{\delta}_{(1,2)}$ is nilpotent and anticommutes with the others δ_i. By virtue of Proposition B.1 already proved for the case $n = 2$, its cohomology is trivial. In order to solve (B.24) we make use of Proposition B.1, true for $(n-1)$, by assumption, where the $(n-1)$ operators are given by $\{\tilde{\delta}_{(1,2)}, \delta_3, \ldots, \delta_n\}$. We have thus the extended form

$$\sum_{\tilde{s},s_3,\ldots,s_n}^{\tilde{s}+s_3+\ldots+s_n=|T|-1} \tilde{\omega}_{(1,2)}^{(\tilde{s},s_3,\ldots,s_n)} = \sum_{s_1,\ldots,s_n}^{s_1+\ldots+s_n=|T|-1} \omega^{(s_1,\ldots,s_n)} \equiv \tilde{\omega}, \tag{B.25}$$

satisfying (B.2):

$$\left(\tilde{\delta}_{(1,2)} + \sum_{i=3}^{n} \delta_i \right) \tilde{\omega} = \tilde{\delta}\, \tilde{\omega} = 0, \quad \tilde{\delta} = \sum_{i=1}^{n} \delta_i.$$

We also know from Part 2 of Proposition B.1 for $(n-1)$ that there exists an extended form

$$\tilde{\varphi} = \sum_{\tilde{s},s_3,\ldots,s_n}^{\tilde{s}+s_3+\ldots+s_n=|T|-2} \tilde{\varphi}_{(1,2)}^{(\tilde{s},s_3,\ldots,s_n)} = \sum_{s_1,\ldots,s_n}^{s_1+\ldots+s_n=|T|-2} \varphi^{s_1,\ldots,s_n} \equiv \tilde{\varphi},$$

which satisfies

$$\tilde{\omega} = \left(\tilde{\delta}_{(1,2)} + \sum_{i=3}^{n} \delta_i \right) \tilde{\varphi} \equiv \tilde{\delta}\, \tilde{\varphi}. \qquad \text{(B.27)}$$

Now we have

Proposition B.2 If the form $\omega = \omega^T = \omega^{(t_1,\ldots,t_n)}$ satisfies

$$\delta_1 \ldots \delta_n \, \omega = 0, \qquad \text{(B.28)}$$

it admits a solution of the type

$$\omega = \sum_{i=1}^{n} \delta_i \, \varphi^{T-E_i}. \qquad \text{(B.29)}$$

The proof of Proposition B.2 is by induction. For the case $n = 1$ (B.28) reads $\delta\, \omega = 0$, and from the triviality of the cohomology of δ the solution is given by

$$\omega = \delta\, \varphi.$$

For the general case, we can rewrite (B.28) as

$$(\delta_1 \ldots \delta_{n-1})\delta_n \, \omega = 0, \qquad \text{(B.30)}$$

and supposing that Proposition B.2 is valid for $(n-1)$, we can solve (B.30) with respect to $\delta_n\omega$, obtaining

$$\delta_n \; \omega^{(t_1,\ldots,t_n-1)} = \sum_{i=1}^{n-1} \delta_i \eta^{T-E_i}. \qquad \text{(B.31)}$$

From Proposition B.1 it follows that

$$\omega = \sum_{i=1}^{n} \delta_i \, \varphi^{T-E_i}. \qquad \text{(B.32)}$$

We have □

Proposition B.3 If the form $\omega^T = \omega^{(t_1,\ldots,t_n)}$ satisfies the following equation:

$$\delta_1 \ldots \delta_{n-1} \, \omega^T + \delta_n \psi^{T+E_1+\ldots+E_{n-1}-E_n} = 0, \qquad \text{(B.33)}$$

then the general solution for it is given by

$$\omega^T = \sum_{i=1}^{n} \delta_i \, \varphi^{T-E_i}. \qquad \text{(B.34)}$$

To proof this we note the following. Applying δ_n on (B.33) we have

$$\delta_1 \ldots \delta_{n-1}\delta_n \; \omega^T = 0,$$

whose solution is (B.34) by virtue of Proposition B.2. □

We have

Proposition B.4 If the form $\omega^T = \omega^{(t_1,\ldots,t_n)}$ satisfies the following set of equations:

$$\delta_i \; \omega^T + \delta_n \psi_i^{T+E_i-E_n} = 0, \quad i = 1, \ldots, n-1, \qquad \text{(B.35)}$$

the general solution is given by

$$\omega^T = \delta_1 \ldots \delta_{n-1} \, \varphi^{T-E_1-\ldots-E_{n-1}} + \delta_n \eta^{T-E_n}. \qquad \text{(B.36)}$$

We also have the following.

Corollary 1 Let $\omega^T = \omega^{(t_1,\ldots,t_n)}$, obey the following set of equations:

$$\delta_i \omega^T = 0, \quad i = 1, \ldots, n. \qquad \text{(B.37)}$$

Then the general solution for (B.37) is

$$\omega^T = \delta_1 \ldots \delta_n \, \varphi^{T-E_1-\ldots-E_n}. \qquad \text{(B.38)}$$

To achieve a proof of Proposition B.4, we write from Proposition B.1 the solution for the first equation of the set (B.35):

$$\omega^T = \delta_1 \varphi^{T-E_1} + \delta_n(\ldots). \qquad \text{(B.39)}$$

Substituting it into the second equation of (B.35) we have

$$\delta_2 \delta_1 \varphi^{T-E_1} + \delta_n(\ldots) = 0. \qquad \text{(B.40)}$$

From Proposition B.3 we get the solution for (B.40):

$$\varphi^{T-E_1} = \delta_1 \varphi^{T-2E_1} + \delta_2 \varphi^{T-E_1-E_2} + \delta_n(\ldots). \qquad \text{(B.41)}$$

Substituting (B.41) into (B.39) we arrive at

$$\omega^T = \delta_1 \delta_2 \varphi^{T-E_1-E_2} + \delta_n(\ldots). \qquad \text{(B.42)}$$

Repeating the argument we finally obtain the result (B.36). □

Appendix C: Truncated extended forms and cohomology

Let us consider as in Appendix B the forms $\phi^R = \phi^{(r_1,\ldots,r_n)}$, which may be fields or superfields. The notation and conventions are explained in the beginning of that appendix. The nilpotent extended operator $\tilde{\delta}$ and an extended q-form of total weight q are defined as

$$\tilde{\delta} = \sum_{i=1}^{n} \delta_i, \qquad \tilde{\phi}^q = \sum_{R}^{|R|=q} \phi^R. \qquad \text{(C.1)}$$

Let us define the truncated extended q forms associated to the highest $H = (h_1, \ldots, h_n)$ (in short: truncated forms) as

$$\check{\phi}^q = \left[\tilde{\phi}^q \right]^{(\mathrm{tr})} = \sum_{R}^{|R|=q,\, R \leq H} \phi^R \qquad \text{(C.2)}$$

The truncation, indicated by the exponent “$^{(\mathrm{tr})}$”, means discarding in the expression all the forms of degree not constrained by $R \leq H$. The polynomials in these truncated forms and their truncated exterior derivatives span the space $\check{\mathcal{E}}_H$, with the exterior multiplication and derivation rules

$$\check{\phi}_1^{q_1}\check{\phi}_2^{q_2} = \left[\tilde{\phi}_1^{q_1}\tilde{\phi}_2^{q_2}\right]^{(\mathrm{tr})}, \quad \check{\delta}\,\check{\phi}^q = \left[\tilde{\delta}\tilde{\phi}^q\right]^{(\mathrm{tr})}. \tag{C.3}$$

$\check{\delta}$ is obviously nilpotent, and the operations defined in (C.3) map $\check{\mathcal{E}}_H$ to $\check{\mathcal{E}}_H$. We have

Proposition C.1 The cohomology of $\check{\delta}$ in the space $\check{\mathcal{E}}_H$ consists of the highest weight truncated forms

$$\check{\phi}^{|H|} = \phi^H, \quad \check{\phi}^{|H|} \neq \check{\delta}\,\check{\phi}^{|H|-1}. \tag{C.4}$$

Let us remark that a highest weight truncated form $\check{\phi}^{|H|}$ is always closed: $\check{\delta}\,\check{\phi}^{|H|} = 0$.

The proof of Proposition C.1 is by induction. The result being obvious for $n = 1$, we shall prove it for the generic case n assuming it to hold for $n-1$. Let us divide the weights $r_1, \dots, r_n$ in two subsets: r_1 and $R' = (r_2, \dots, r_n)$. We define accordingly the partially extended operator $\tilde{\delta}'$ and the partially extended forms

$$\tilde{\delta}' = \sum_{i=2}^{n}\delta_i, \qquad \tilde{\phi}'^{\,(q-r,r)} = \sum_{R'}^{|R'|=r}\phi^{(q-r,R')}, \quad r = 0, \dots, q, \tag{C.5}$$

as well as the partially truncated extended forms

$$\check{\phi}'^{\,(q-r,r)} = \left[\tilde{\phi}'^{\,(q-r,r)}\right]^{(\mathrm{tr}')} = \sum_{R'}^{|R'|=r,\,R'\le H'}\phi^{(q-r,R')}, \qquad r = 0, \dots, q, \tag{C.6}$$

on which act the partially truncated derivative $\check{\delta}'$ defined by

$$\begin{aligned}\check{\delta}'\check{\phi}'^{\,(q-r,r)} &= \left[\tilde{\delta}'\tilde{\phi}'^{\,(q-r,r)}\right]^{(\mathrm{tr}')} \\ &= \sum_{i=2}^{n}\left(\sum_{R'}^{|R'|=r,\,R'\le H'-E_i}\delta_i\phi^{(q-r,R')}\right), \\ &\qquad r = 0, \dots, q.\end{aligned} \tag{C.7}$$

From the induction hipothesis, the cohomology of $\check{\delta}'$ is trivial in the subspace of the partially truncated forms (C.7) restricted by the condition $r < h'$[8].

Let us now solve the cohomology equation:

$$\check{\delta}\,\check{\phi}^q = 0. \tag{C.8}$$

The truncated form (C.2), can be written as

$$\check{\phi}^q = \sum_{r=\mathrm{Max}(0,q-h_1)}^{\mathrm{Min}(q,h')}\check{\phi}'^{\,q-r,r}. \tag{C.9}$$

We thus have to examine separately the four cases $0 \le q < h_1$, $h_1 \le q < h'$, $h' \le |H| < h_1$, $q = |H| = h_1 + h'$,

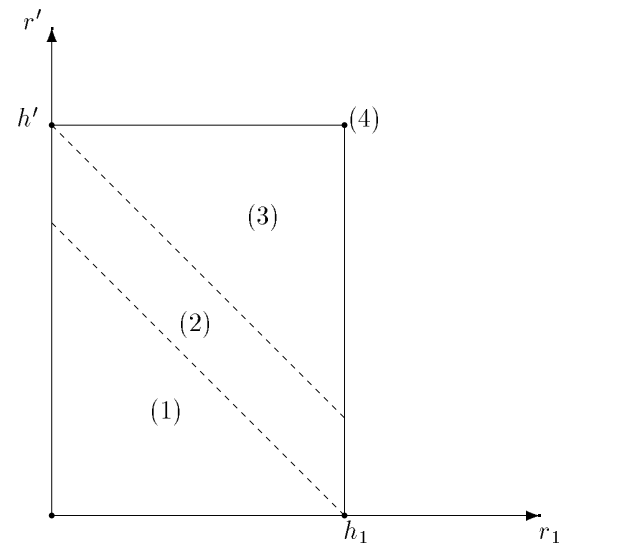

Fig. C.1. Weight diagram for the partially truncated superforms $\check{\phi}'^{\,(r_1,r')}$. The numbers (1), (2), (3) and (4) refer to the four cases examined in the text

corresponding respectively to the areas (1), (2), (3) and to the point (4) of Fig. C.1.

Case 1: $0 \le q < h_1$. We have

$$\check{\phi}^q = \sum_{r=0}^{q}\check{\phi}'^{\,(q-r,r)}, \qquad \check{\delta}\,\check{\phi}^q = \sum_{r=0}^{q}(\delta_1 + \check{\delta}')\check{\phi}'^{\,(q-r,r)}.$$

The cohomology condition (C.8) implies the following equations:

$$\begin{aligned}&\delta_1\check{\phi}'^{\,(q,0)} = 0, \\ &\check{\delta}'\check{\phi}'^{\,(q-r,r)} + \delta_1\check{\phi}'^{\,(q-r-1,r+1)} = 0, \quad r = 1, \dots, q-1, \\ &\check{\delta}'\check{\phi}'^{\,(0,q)} = 0.\end{aligned}$$

Solving these equations in turn, beginning from the first one, we obtain easily, using the triviality of the cohomology of δ_1,

$$\begin{aligned}&\check{\phi}'^{\,(q,0)} = \delta_1\check{\psi}'^{\,(q-1,0)}, \\ &\check{\phi}'^{\,(q-r,r)} = \delta_1\check{\psi}'^{\,(q-r-1,r)} + \check{\delta}'\check{\psi}'^{\,(q-r,r-1)}, \\ &\qquad r = 1, \dots, q-1, \\ &\check{\phi}'^{\,(0,q)} = \check{\delta}'\check{\psi}'^{\,(0,q-1)}.\end{aligned}$$

This result can be rewritten as

$$\check{\phi}^q = \check{\delta}\,\check{\psi}^{q-1}, \quad \text{with} \quad \check{\psi}^{q-1} = \sum_{r=0}^{q-1}\check{\psi}'^{\,(q-1-r,r)}. \tag{C.10}$$

We note that for $q = 0$ we have $\check{\phi}^0 = \check{\phi}'^{\,(0,0)}$ and the solution is $\check{\phi}^0 = 0$. We have thus proven the triviality of the cohomology in case 1.

[8] We use the notation h' for $|H'| = \sum_{i=2}^{n} h_i$.

Case 2: $h_1 \leq q < h'$. In this case the truncated form is given by

$$\check{\phi}^q = \sum_{r=q-h_1}^{q} \check{\phi}'^{\,(q-r,r)},$$

and the cohomology condition (C.8) yields

$$\begin{aligned} &\check{\delta}'\check{\phi}'^{\,(h_1-r,q-h_1+r)} + \delta_1\check{\phi}'^{\,(h_1-r-1,q-h_1+r+1)} = 0, \\ &\qquad r = 0,\ldots,h_1, \\ &\check{\delta}'\check{\phi}'^{\,(0,q)} = 0. \end{aligned}$$

This time one has to begin with the last of these equations, and use the induction hypothesis according to which the cohomology of $\check{\delta}'$ is trivial when applied to partial truncated forms (C.6) which are not of maximal weight, i.e such that $r < h'$. The solution reads

$$\begin{aligned} &\check{\phi}'^{\,(0,q)} = \check{\delta}'\check{\psi}'^{\,(0,q-1)}, \\ &\check{\phi}'^{\,(r,q-r)} = \check{\delta}'\check{\psi}'^{\,(r,q-r-1)} + \delta_1\check{\psi}'^{\,(r-1,q-r)}, \\ &\qquad r = 1,\ldots,h_1, \end{aligned}$$

which again may be written as in (C.10), showing the triviality of the cohomology in case 2.
Case 3: $h' \leq q < |H|$. The truncated form reads

$$\check{\phi}^q = \sum_{r=q-h_1}^{q-h'} \check{\phi}'^{\,(q-r,r)},$$

and the cohomology condition (C.8) yields

$$\begin{aligned} &\check{\delta}'\check{\phi}'^{\,(h_1-r,q-h_1+r)} + \delta_1\check{\phi}'^{\,(h_1-r-1,q-h_1+r+1)} = 0, \\ &\qquad r = 0,\ldots,h_1+h'-q-1. \end{aligned} \tag{C.11}$$

The situation is a bit more subtle. We begin from (C.11) with $r = 0$, and use the result of Corollary 2 of Proposition B.1 – valid due to the triviality of the cohomology of both δ_1 and $\check{\delta}'$ –, from which we can write

$$\begin{aligned} \check{\phi}'^{\,(h_1,q-h_1)} &= \check{\delta}'\check{\psi}'^{\,(h_1,q-h_1-1)} + \delta_1\check{\psi}'^{\,(h_1-1,q-h_1)}, \\ \check{\phi}'^{\,(h_1-1,q-h_1+1)} &= \check{\delta}'\check{\psi}'^{\,(h_1-1,q-h_1)} + \delta_1\check{\psi}'^{\,(h_1-2,q-h_1+1)}. \end{aligned} \tag{C.12}$$

Now substituting this result in (C.11) for $r = 1$ and using the triviality of the cohomology of δ_1, we get the first of the following equations – the one for $r = 2$ is

$$\begin{aligned} &\check{\phi}'^{\,(h_1-r,q-h_1+r)} \\ &= \check{\delta}'\check{\psi}'^{\,(h_1-r,q-h_1+r-1)} + \delta_1\check{\psi}'^{\,(h_1-r-1,q-h_1+r)}, \\ &\qquad r = 2,\ldots,h_1+h'-1, \end{aligned} \tag{C.13}$$

and the remaining ones, for $r \geq 3$, are obtained in the usual way using the triviality of the cohomology of δ_1. The result (C.13) can be rewritten as (C.10), showing the triviality of the cohomology in case 3.

Case 4: $q = |H| = h_1 + h'$ This is the case of highest weight:

$$\check{\phi}^q = \check{\phi}'^{\,(h_1,h')} = \phi^{(h^1,\ldots,h^n)},$$

which satisfies identically the cohomology condition (C.8). It may be the $\check{\delta}'$-variation of some truncated form $\check{\psi}'^{\,|H|-1}$, or not. In the latter case it belongs to the cohomology of $\check{\delta}'$.

Joining together the results of these four cases ends the proof of Proposition C.1. □

Appendix D: Wess–Zumino gauge and covariant superfields

As shown in [10] it is possible to fix algebraically the gauge degrees of freedom corresponding to the ghosts $c_{I_1\ldots I_n}(x)$ $(1 \leq n \leq N)$, through the conditions[9]

$$e_I(x) = 0, \quad e_{[II_1\ldots I_n]}(x) = 0 \ (1 \leq n \leq N). \tag{D.1}$$

This defines the so-called Wess–Zumino (WZ) gauge, analogous to the one encountered in the supersymmetric Yang–Mills theories [13]. We are left with the usual gauge degree of freedom corresponding to the ghost $c(x)$. The physical degrees of freedom are labeled by the covariant fields

$$\left\{ F_a,\, a_{I_1\ldots I_n},\, e^{(\mathrm{M})}_{I_1\ldots I_{n+1}},\, D_a a_{I_1\ldots I_n},\, D_a e^{(\mathrm{M})}_{I_1\ldots I_{n+1}}\,;\, n \geq 1 \right\}, \tag{D.2}$$

where F_a and D_a are the Yang–Mills curvature and the covariant exterior derivative with respect to the connection a, and $e^{(\mathrm{M})}_{I_1\ldots I_n}$ is the mixed symmetry tensor corresponding to the second Young tableau in the right-hand side of Fig. D.1, and defined by the expansion

$$e_{I_1I_2\ldots I_{n+1}} = e_{[I_1I_2\ldots I_{n+1}]} + e^{(\mathrm{M})}_{I_1I_2\ldots I_{n+1}},$$

with

$$e^{(\mathrm{M})}_{I_1I_2\ldots I_{n+1}} = \frac{1}{n+1}\sum_{k=2}^{n+1}\left(e_{I_1I_2\ldots I_k\ldots I_{n+1}} + e_{I_kI_2\ldots I_1\ldots I_{n+1}}\right). \tag{D.3}$$

The BRST transformations of the fields (D.2) are covariant:

$$\mathcal{S}\phi = -[c,\phi]. \tag{D.4}$$

The stability of the WZ-gauge choice requires a redefinition of the SUSY operators – acting on the fields (D.2) and a:

$$\tilde{Q}_I = Q_I + \delta_{(\lambda_I)}, \tag{D.5}$$

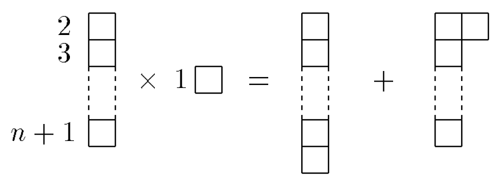

Fig. D.1. Expansion (D.3) of the tensor $e_{I_1I_2\ldots I_n}$

[9] The field components of the superfields of the theory are defined by (2.9) and (2.10).

where $\delta_{(\lambda_I)}$ is a supergauge transformation of field dependent parameters

$$\lambda_{II_1\ldots I_n} = \frac{1}{n!n} e^{(\mathrm{M})}_{II_1\ldots I_n}, \tag{D.6}$$

equivalent to a superfield BRST transformation (2.12) with the ghost components $c_{I_1\ldots I_n}$ given by (D.6) and with $c = 0$ [10]. The superalgebra (2.1) now closes on the gauge transformations

$$[\tilde{Q}_I, \tilde{Q}_J] = -2\delta_{\mathrm{gauge}}(e^{(\mathrm{M})}_{IJ}) \tag{D.7}$$

of the field dependent parameters $e^{(\mathrm{M})}_{IJ}$.

Let us now show that there is a bijection between the set of fields (D.2) and the following set of covariant superfields:

$$\left\{ F_A, \Psi^{(\mathrm{A})}_{I_1\ldots I_n}, \Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}, D_A\Psi^{(\mathrm{A})}_{I_1\ldots I_n}, D_A\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}\,;\, n \geq 1 \right\}. \tag{D.8}$$

F_A and D_A are the Yang–Mills curvature and the covariant exterior derivative with respect to the superfield connection A, and the remaining elements are defined from the supercurvature components (2.14) by

$$\Psi^{(\mathrm{A})}_{I_1\ldots I_n} = \Psi_{[I_1|I_2\ldots I_n]}, \tag{D.9}$$
$$\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}} = \frac{n}{n+1}\left(\Phi_{I_1[I_2|I_3\ldots I_{n+1}]} + \Phi_{I_2[I_1|I_3\ldots I_{n+1}]}\right).$$
$$n \geq 1. \tag{D.10}$$

The bracket $[\ldots]$ means complete antisymmetrization in the indices, the bar $|$ symbolizes covariant θ-derivations,

$$\Psi_{I_1|I_2\ldots I_n} = D_{I_2}\ldots D_{I_n}\Psi_{I_1}, \quad D_I X = \partial_I X + [E_I, X],$$

and the mixed symmetry tensors $\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}$ belong to the expansion of the covariant θ-derivatives $\Phi_{I_1I_2|I_3\ldots I_{n+1}}$ in irreducible representations of the permutation group S_{n+1}: they correspond to the Young diagram shown in the right-hand side of Fig. D.2.

All the objects X in (D.8) are covariant, i.e.

$$\mathcal{S}X = -[C, X].$$

The bijection between the set (D.2) and the set (D.8) is simply given by the fact that the elements of the former are equal to the $\theta = 0$ components of the elements of the latter, provided the WZ-gauge conditions (D.1) are applied:

$$\left(F_a, a_{I_1\ldots I_n}, e^{(\mathrm{M})}_{I_1\ldots I_{n+1}}, D_a a_{I_1\ldots I_n}, D_a e^{(\mathrm{M})}_{I_1\ldots I_{n+1}}\right) =$$
$$\left(F_A, \Psi^{(\mathrm{A})}_{I_1\ldots I_n}, \Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}, D_A\Psi^{(\mathrm{A})}_{I_1\ldots I_n}, D_A\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}\right)\Big|_{\theta=0,\mathrm{WZ}},$$

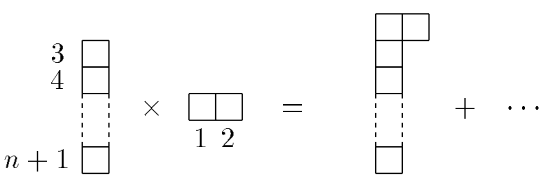

Fig. D.2. Expansion of the tensor $\Phi_{I_1I_2|I_3\ldots I_{n+1}}$

This is obvious for F_A. For $\Psi^{(\mathrm{A})}_{I_1\ldots I_n}$, we observe that each θ-derivative of Ψ_I brings in factors of θ-derivatives $E_{I,I_1,\ldots I_k}$ or space-time covariant derivatives of them. However, only completely antisymmetrized derivatives $E_{[I,I_1,\ldots I_k]}$ may contribute as factors to the completely antisymmetric tensor $\Psi^{(\mathrm{A})}$. Since these completely antisymmetrized derivatives vanish at $\theta = 0$ due to the WZ-gauge conditions, we are left with the simple θ-derivatives of A, which at $\theta = 0$ yield the fields $a_{I_1\ldots I_n}$. The same conclusion holds for the terms $D_A\Psi^{(\mathrm{A})}_{I_1\ldots I_n}$.

The argument is similar for the terms $\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}$ (and $D_A\Phi^{(\mathrm{M})}_{I_1\ldots I_{n+1}}$): only factors of completely antisymmetrized derivatives of E_I may contribute to these mixed symmetry tensors, made from covariant θ-derivatives of the *symmetric* tensor $\Phi_{I_1I_2}$ and symbolized by the diagram shown in the right-hand side of Fig. D.2. And this same diagram defines the symmetry properties of $e^{(\mathrm{M})}_{I_1\ldots I_{n+1}}$.

References

1. E. Witten, Commun. Math. Phys. **117**, 353 (1988); Int. J. Mod. Phys. A **6**, 2775 (1991)
2. L. Baulieu, I.M. Singer, Nucl. Phys. B (Proc. Suppl.) B **5**, 12 (1988)
3. D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys. Rev. **209**, 129 (1991)
4. S. Donaldson, J. Diff. Geom. **30**, 289 (1983); Topology **29**, 257 (1990)
5. J.P. Yamron, Phys. Lett. B **213**, 325 (1988)
6. D. Birmingham, M. Blau, G. Thompson, Int. J. Mod. Phys. A **5**, 4721 (1990); M. Blau, G. Thompson, Commun. Math. Phys. **152**, 41 (1993) [hep-th/9112012]; Nucl. Phys. B **492**, 545 (1997) [hep-th/9612143]
7. S. Marculescu, Nucl. Phys. B (Proc. Suppl.) B **56**, 148 (1997); H.D. Dahmen, S. Marculescu, T. Portmann, Nucl. Phys. B **462**, 493 (1996) [hep-th/9506162]
8. N.R.F. Braga, C.F.L. Godinho, Phys. Rev. D **61**, 125019 (2000) [hep-th/9905003]
9. B. Geyer, D. Mulsch, Nucl. Phys. B **616**, 476 (2001) [hep-th/0108042]; B **662**, 531 (2003) [hep-th/0211061]
10. C.P. Constantinidis, O. Piguet, W. Spalenza, Eur. Phys. J. C **33**, 443 (2004) [hp-th/0310184]
11. Wesley Spalenza, Fixações de gauge para o modelo super-BF, PhD thesis, Centro Brasileiro de Pesquisa Físicas – CBPF (2004)
12. J.H. Horne, Nucl. Phys. B **318**, 2234 (1989)
13. S.J. Gate, M.T. Grisaru, M. Roček, W. Siegel, Superspace, Frontiers of Physics, v. 58 (1983)
14. J.L. Boldo, C.P. Constantinidis, F. Gieres, M. Lefrançois, O. Piguet, Int. J. Mod. Phys. A **19**, 2971 (2004) [hep-th/0303053]; A **18**, 2119 (2003) [hep-th/0303084]
15. P. van Baal, S. Ouvry, R. Stora, Phys. Lett. B **220**, 159 (1989)
16. N. Maggiore, O. Piguet, S. Wolf, Nucl. Phys. B **458**, 403 (1996); Erratum B **469**, 513 (1996) [hep-th/9507045]
17. O. Piguet, S.P. Sorella, Algebraic renormalization (Springer-Verlag 1995)
18. G. Barnich, F. Brandt, M. Henneaux, Phys. Rep. **338**, 439 (2000)